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THE STABILITY OF THE STEADY - STATE SOLUTIONS IN THE THEORY OF THERMAL EXPLOSIONS

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The steady-state theory of thermal explosion is concerned with solutions of the following boundary value problem [1 and 2]:

$$\Delta T + \varphi(T) = 0, \quad T|_\Gamma = 0 \tag{0.1}$$

where Γ is the surface enclosing the region G (a vessel), T is the temperature and $\varphi(T)$ is a positive, monotonously increasing function differentiable on $[0, +\infty]$. If a solution of (0.1) exists, then we assume that no explosion takes place in the vessel G, otherwise we assume that it does occur. In [1 and 2] the problem was studied for $\varphi(T) = e^T$ and the regions which possessed plane, cylindrical or spherical symmetry. In accordance with this the problem can be reduced to a problem for a segment, circle or a sphere with one independent variable equal to the distance from the center.

For a segment, the problem (0.1) becomes

$$\frac{d^2T}{dx^2} + \varphi(T) = 0, \quad T|_{x=\pm h} = 0 \tag{0.2}$$

In [1 and 2] it was shown that a critical value $h = h_*$ exists for $\varphi(T) = e^T$ such, that when $0 \leqslant h \leqslant h_*$, a solution of (0.2) exists. When $h > h_*$, we have no solution, while when $0 < h < h_*$, we actually have two solutions. Denoting by $T_m = T(0)$ the maximum temperature and introducing the function $h = h(T_m)$, we obtain the corresponding curve as shown on Fig. 1. By symmetry we have $dT/dx = 0$ when $x = 0$ and the function $h(T_m)$ is singlevalued and continuous (a solution of the Cauchy's problem for (0.2) with conditions $dT/dx = 0$ and $T = T_m$ when $x = 0$ exists and depends continuously on T_m). In the case of a circle we have the analogous result. In the case of a sphere, a critical value of the radius exists also, but according to [2] the curve $h(T_m)$ is more complex. In [3] the problem was investigated for a function $\varphi(T)$ of the sufficiently general form and it was found that more than two solutions may exist for a given h, although uniqueness is not excluded. The curve $h(T_m)$ may have

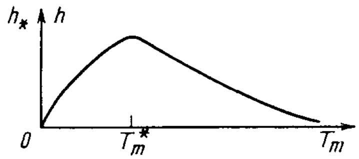

Fig. 1

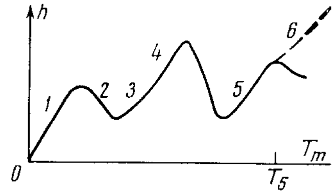

Fig. 2

several maxima (Fig. 2) and its shape depends on $\varphi(T)$.

If $\lim T^{-1}\varphi(T) = 0$ when $T \to \infty$, then $h(T_m) \to \infty$ as $T_m \to \infty$. If $\lim T^{-1}\varphi(T) = A$ when $T \to \infty$ where $0 < A < \infty$, then $\lim h(T_m)$ when $T_m \to \infty$, will be a finite, positive quantity. If $\lim T^{-1}\varphi(T) = \infty$ when $T_m \to \infty$, then $h(T_m) \to 0$ as $T_m \to \infty$.

In [4], the stability of steady-state solutions for $\varphi(T) = e^T$ was investigated for a number of symmetric regions. It was shown that only these solutions are stable, for which $T_m < T_m^*$ where $h(T_m^*) = h_*$. In [5], some problems of stability for regions of arbitrary shape and for the functions $\varphi(T)$ for which $\varphi''(T) > 0$, were investigated. In particular it was shown that there exist such small regions, for which small solutions are stable and if a stable solution exists for a region G, then it exists for any $G' \subseteq G$.

The present paper considers the stability for the arbitrary functions $\varphi(T)$. We assume that $\varphi'(T)$ is bounded on the arbitrary, but finite interval. Section 1 deals with the case of symmetric regions, Section 2 repeat it using another method and Section 3 generalizes the results to arbitrary regions.

1. Let us consider the problem (0.2) for a segment. In the case of a circle and a sphere, the arguments used are identical. Let us subdivide the curve $h(T_m)$ into intervals corresponding to the monotonous variation of the function (Fig. 2) and let us consider, for each segment, a family of solutions $T(x, h)$ continuous and dependent on h. We easily see that $T^\cdot = dT/dh$ satisfies

$$d^2T^\cdot/dx^2 + \varphi'(T)\,T^\cdot = 0 \tag{1.1}$$

and the condition

$$dT^\cdot / dx = 0 \qquad \text{for } x = 0 \tag{1.2}$$

Any solution $\psi(x)$ of the problem

$$\frac{d^2\psi}{dx^2} + \varphi'(T)\,\psi = 0, \qquad \frac{d\psi}{dx}\bigg|_{x=0} = 0 \tag{1.3}$$

has the form $\psi = cT^\cdot$ where c is a constant, and the converse is true. Therefore, if $T^\cdot$ becomes zero within the interval $(-h, h)$, then ψ also becomes zero and vice-versa.

Using the method of small perturbations to study the stability of the steady-state solution, we arrive at the problem of determination of the sign of the smallest eigenvalue λ

$$d^2\psi / dx^2 + \varphi'(T)\,\psi = -\lambda\psi, \quad \Psi|_{x=\pm h} = 0 \tag{1.4}$$

We find that if the sign is negative, the solution $T(x, h)$ is unstable, while the positive sign implies stability. We know [6] that the smallest eigenvalue has an associated eigenfunction which does not become zero within the considered interval and, that the increase in the length of the interval leads to the decrease in the smallest eigenvalue.

Investigation of stability can be conveniently connected with the concept of an envelope of the family $T(x, h)$ of integral curves of (0.2) corresponding to the continuous branch of solutions. We shall, in fact, show that, if the family has an envelope then every tangent to that envelope corresponds to an unstable solution. Conversely, if for some h_0 the solution is unstable, then an envelope of $T(x, h)$ exists for h sufficiently close to h_0.

Indeed, let the family $T(x, h)$ possess an envelope tangent to the curve $T(x, h_0)$ at the point $x_0 (0 < x_0 < h_0)$, then a solution of (1.1) exists, for which $T^\cdot = 0$ when $x = \pm x_0$. Therefore a solution of (1.3) exists, for which $\lambda = 0$ and which satisfies the condition $\psi|_{x=\mp x_0} = 0$

when $x = x_0$ and a solution of (1.4) exists for some $\lambda < 0$, i.e. the considered solution is unstable. Conversely we may assume that the solution $T(x, h_0)$ is unstable, and this will imply that a solution of (1.4) exists for which the first eigenvalue $\lambda_0 < 0$.

Since the first eigenvalue increases with decreasing length of the interval and becomes positive at sufficiently small values of the intervals, such a point x_0, $0 < x_0 < h_0$ can be found by virtue of the fact that the first eigenvalue depends continuously on the interval length, that for the interval $(-x_0, x_0)$ the first eigenvalue will be equal to zero. But since $T^{\cdot} = c\psi$, we have $T^{\cdot} = 0$ when $x = \pm x_0$.

Further we find that, since the eigenvalue is dependent continuously on the coefficients and on the length of the interval, it follows that if a solution of (1.4) exists which has $\lambda_0 < < 0$ for $h = h_0$, then for h almost equal to h_0 the problem (1.4) also has negative first eigenvalues and functions $T^{\cdot}(x, h)$ will become zero for h specified above, at the points x lying in some vicinity of x_0. This in turn implies that the corresponding family $T(x, h)$ has an envelope.

Usefulness of this assertion becomes clear, when it is applied to the case $\varphi(T) = e^T$ which was considered in [4]. The results obtained in [4] with much effort, follow from our assertion directly. Authors of [4] used the envelope equation in their proof and noted this fact.

Our assertion facilitates the investigation in the cases when the existence or nonexistence of an envelope can be deduced from other (e.g. geometrical) considerations. For example, a family of convex curves enclosing each other in turn has no envelope. It is of interest, that the author of [2] based his proof of the stability of the steady-state solution in the theory of combustion on the fact, that the family $T_0\ (x + c)$ has no envelope.

It can easily be shown that the segments over which the function $h(T_m)$ decreases, correspond to unstable solutions. Indeed, in this case the curves $T(x, h)$ for different h intersect, and for each admissible h a point $0 < x_h < h$ exists for which $T^{\cdot}(x_h, h) = 0$. Obviously, if $T^{\cdot}(x, h)$ preserved its sign for the given value of h at all x, it would mean that $T(x, h)$ is monotonous under the variation of h for all x.

Let us now consider the segments over which $h(T_m)$ increases. First we shall consider the first increasing segment which begins at the coordinate origin and corresponds to solutions with smaller values of T_m. We shall show that a stable solution corresponds to this segment. Function $T(x, h)$ satisfies the following integral equation

$$T(x, h) = \int_{-h}^{h} K_h(x, \xi)\, \varphi\,[T(\xi)]\, d\xi \tag{1.5}$$

where $K_h(x, \xi)$ is the corresponding Green's function.

It can easily be shown that the solution with the smallest T_m can be obtained as the limit of the following sequence when $k \to \infty$

$$T_k(x, h), \qquad k = 0, 1, 2, \ldots, \qquad T_0 = 0 \qquad \left(T_k(x, h) = \int_{-h}^{h} K_h(x, \xi)\, \varphi\,[T_{k-1}(\xi)]\, d\xi\right)$$

Since $K_{h1} < K_{h2}$ when $h_1 < h_2$, we have $T(x, h_1) \leqslant T(x, h_2)$. However, a stronger inequality $T(x, h_1) < T(x, h_2)$ exists. Indeed, if at some point we had $T(x_0, h_1) = T(x_0, h_2)$, then the corresponding (nonintersecting) curves would have a common tangent and the solutions would coincide. Thus the family $T(x, h)$ corresponding to the first segment over which $h(T_m)$ increases, constitutes a family of convex curves $(d^2T/dx^2 < 0)$ enclosing each other in turn, which has no envelope, and this corresponds to stable solutions.

This proof is valid for any segment of the curve with the smallest T_m (segment *4* of Fig. 2).

It remains to show the stability of solutions for the increasing segments *3* and *5* on Fig. 2. We shall show that if, on moving along a branch of the curve $h(T_m)$ we pass from the stable to the unstable solutions, then a point exists on this branch, for which $dh/dT_m = 0$. Let us reformulate the problem (0.2) in the following manner. Introducing an independent variable $y = x/h$, we obtain

$$d^2T / dy^2 + h^2\varphi(T) = 0, \qquad T|_{y=\pm 1} = 0 \tag{1.6}$$

and the problem of stability reduces to the problem

$$d^2\psi / dy^2 + h^2\varphi'(T)\psi = -\lambda\psi, \quad \Psi|_{y=\pm 1} = 0 \tag{1.7}$$

Let us suppose that during a continuous variation of h along the given branch, a transition from stability to instability occurs. Then, since the first eigenvalue is continuously dependent on h and should change its sign, there exists such a value of h for which the problem has a solution in which $\lambda = 0$ and which does not become zero within the interval $(-1, 1)$. We shall denote this solution by $\psi_0(y)$. Let us now differentiate (1.6) with respect to T_m. Then, putting $T' = dT/dT_m$, we obtain

$$\frac{d^2T}{dy^2} + h^2\varphi'(T)T' = -2h\frac{dh}{dT_m}\varphi(T), \qquad T'|_{y=\pm 1} = 0 \tag{1.8}$$

Since the homogeneous boundary value problem has the solution $\psi_0(y)$, the nonhomogeneous problem can have a solution only if the condition

$$\frac{dh}{dT_m}\int_{-1}^{1}\psi_0(y)\varphi(T)\,dy = 0$$

is fulfilled. Consequently $dh/dT_m = 0$ and the segment *3* of Fig. 2 gives stable solutions, since the segment *4*, as we know, has stable solutions.

To prove the stability of solutions on the segment *5* of Fig. 2, we shall proceed as follows. We can extend the function $\varphi(T)$ for the values $T > T_5$ (which is fixed) so, that $h(T_m)$ increases when $T > T_5$ (broken line *6* on Fig. 2), by putting $\varphi(T) = \varphi(T_5)$ for $T > T_5$ and rounding the region of contact so that it is convex.

Since the form of $\varphi(T)$ does not, at high values, influence the solutions with smaller values of T, we infer that the solutions are stable on the segment *5*. Thus we have shown that the stable branches of solutions correspond to all increasing segments. A direct consequence of this is, that if the problem (0.2) has several solutions for a given h, then the first solution is stable, the second one unstable, etc. alternately, with the solutions arranged in the order of increasing T_m. Obviously, the above reasoning can be extended to the case of a circle and a sphere without substantial alterations.

2. We shall now investigate the stability using another method, which may be found useful in solving problems of this type.

It can easily be shown that the solution which tends to zero when $h \to 0$, is stable. The proof follows that of [5] with the additional condition that $\varphi'(T)$ is bounded when $T \to 0$.

Then, multiplying (1.7) by T' and (1.8) by ψ, subtracting the second from the first and integrating the result over the interval $[-1, 1]$, we obtain

$$\frac{\lambda}{h'} = \int_{-1}^{1}\psi\varphi(T)\,dy \Big/ \int_{-1}^{1}\psi T'\,dy \tag{2.1}$$

Let λ be the smallest eigenvalue of the problem (1.7); then from (2.1) it follows that if $h' > 0$ and $T' \geqslant 0$, the $\lambda > 0$ and the first solution is stable.

Let now $h' \to 0$. When $h' = 0$, Eq. (1.7) has the solution $\psi = T'$, therefore (2.1) yields

$$\lim_{h'\to 0}\frac{\lambda}{h'} = \int_{-1}^{1}T'\varphi(T)\,dy \Big/ 2h\int_{-1}^{1}T'^2\,dy$$

As we have $T' \geqslant 0$ for the smallest solution, the integral on the right-hand side is positive. This in turn implies that $d\lambda/dh' > 0$ when $h' = 0$ and, that λ changes its sign together with h'. Combining this with the results of Section 1 we easily come to the conclusion that the solutions are stable over the increasing segments of the curve $h(T_m)$ and unstable over the decreasing ones.

3. The above results can be generalized to the case of arbitrary regions. Let us consider the solution of (0.1) for an arbitrary region. In addition to (0.1) we shall consider the corresponding integral equation

$$T = \int_G K(P, Q) \varphi [T(Q)] \, dQ$$

Let us now introduce a parameter h and consider the resulting equation

$$T = h \int_G K(P, Q) \varphi [T(Q)] \, dQ \tag{3.1}$$

Results of [7] imply that this equation has a continuous, infinite branch of solutions in the space of continuous functions. Using the length R of the solution as a parameter defining the solution of (3.1) and introducing the function $h(R)$ analogous to $h(T_m)$, we can extend the results of Sections 1 and 2 to the case of an arbitrary region without significant changes. At the same time we obtain the results of [8] in a slightly stronger form, namely that when the region increases, the smallest eigenvalue decreases. If several solutions exist for a given h, then on arranging them in the order of increasing length we find, that the first solution is stable, the second one unstable, etc. alternately. In this case we can take max $T(P)$ with $P \in G$ as the measure of the length R.

The methods given above can be used in investigating the stability of solutions of more general equations with different boundary conditions. We can, for example, apply them to equations of the type $L(T) + \varphi(T) = 0$ where $L(T)$ is a selfconjugate operator.

In conclusion we note that the problem on the thermal stability of steady-state solutions can be studied using the same methods, when viscous fluid flows in a two-dimensional channel and in a circular tube are investigated, with heat generated by friction and the dependence of viscosity on temperature [9] both taken into account.

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